Stars & galaxies, example class 4

Short questions

1. Evidence for the presence of a very dark, massive object, presumably a black hole, comes from, for example,
   - photometric observations, suggesting a high density of stars
   - stellar kinematics, implying very high velocities near the centre
   - measurements of maser emission, suggesting a very high mass-density.
   - (several others are mentioned in the notes)
2. The evidence is based on proper-motion studies of stars close to the centre, which imply the presence of a very massive but dark object.
3. The presence of the clusters causes the images of background galaxies to be distorted, or images of the same background galaxy to appear several times. Modelling of the required potential constrains the mass of the cluster.

Question 1.

- Since $N(M)\,dM$ is the number of stars with mass between M and $M + dM$, the total mass of stars in that interval is $M\,N(M)\,dM$, and therefore, expressing all masses in $M_\odot$, the total mass of stars in the cluster is
  $$M_T = \int_{M_-}^{M_+} M\,N(M)dM = \int_{M_-}^{M_+} M^{-1.35}\,dM = 5.5\,N_0 \qquad (1)$$
  hence $N_0 = 1.d6/5.5$ (in which $M_- = 0.1$ and $M_+ = 30$. The number of SNe is
  $$N = \int_{8}^{M_+} N(M)\,dM = 0.04\,N_0 = 6700\,. \qquad (2)$$
- In solar units, the total luminosity is
  $$L_T = \int_{M_-}^{M_+} M^2\,N(M)dM = 13.7\,N_0 \qquad (3)$$
  therefore the mass-to-light ratio is $M_T/L_T = 0.4$.

- In this case we need to use $1M_\odot$ for M_+, hence the luminosity is now

$$L_T = \int_{M_-}^{1} M^2\, N(M) dM = 1.2\, N_0 \tag{4}$$

therefore the mass-to-light ratio is $M_T/L_T = 3..$

Question 2.
The equation for the angle, θ_E, under which you see the ring is

$$\begin{aligned} \theta_E &= \frac{R_E}{D_{OL}} \\ &= (\frac{4GM}{c^2})^{1/2} \left(\frac{D_{OS} - D_{OL}}{D_{OS}\, D_O L}\right)^{1/2} \end{aligned} \tag{5}$$

Substituting the numbers ($M_\odot = 1.989 \times 10^{30}$kg, 1pc $= 3.086 \times 10^{16}$m, G $= 6.67 \times 10^{-11}\,$m^{3}kg^{-1}s^{-2}, $c = 3 \times 10^8$ms^{-1}), this becomes

$$\begin{aligned} \theta_E &= [\frac{4 \times 6.67 \times 1.989}{3.086 \times 3.^2} \times \frac{10^{-11} \times 10^{13} \times 10^{30}}{10^3 \times 10^6 \times 10^{16} \times 10^{16}} \times \frac{\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}\mathrm{kg}}{\mathrm{m\,m}^2/\mathrm{s}^{-2}}]^{1/2} \\ &= 4.37110 \times 10^{-5}\,\text{radians} \end{aligned} \tag{6}$$

which is 9.0 arc seconds. Note how it is easy to get the order of magnitude by just working-out the exponents, and how you can check whether the answer makes sense by checking the units.

The angle under which you see the galaxy is

$$\theta = \frac{20\text{kpc}}{500\text{Mpc}} = 8.25\text{arcsec} \tag{7}$$

The diameter of the ring is slightly larger than that of the galaxy.

Question 3.

- The mass due to the density distribution $\rho(r)$ is

$$M_\rho(< r_1) = 4\pi\rho_1 \int_0^r \frac{r_1}{r}\, r^2\, dr = 2\pi\rho_1 r_1 r^2 = M_1 (r/r_1)^2 \qquad (8)$$

for $r < r_1$, where $M_1 \equiv 2\pi\rho_1 r_1^3$. (Note that this is found by summing the masses of spherical shells, $M(< r) = \int_0^M dM$, where the mass of a spherical shell with radius r, density $\rho(r)$ and thickness dr is $dM = 4\pi r^2 \rho(r) dr$)

The mass in a shell between r_1 and $r < r_2$ is

$$M_\rho(r_1 < r) = 4\pi\rho_1 \int_{r_1}^r (r_1/r)^2\, r^2\, dr = 2M_1(r/r_1) - 2\, M_1 = 2M_1(x-1)\,, \qquad (9)$$

hwere $x \equiv r/r_1$. Therefore, the enclosed mass is

$$\begin{aligned} M(< r) &= M_0 + M_1 x^2 \quad \text{for } r \le r_1 \\ &= M_0 + 2\, M_1\, (x_1) \quad \text{for } r_1 \le r \le r_2 \\ &= M_0 + 2\, M_1\, (r_2/r_1 - 1) \;\; \text{for } r > r_2\,. \end{aligned} \qquad (10)$$

- The circular velocity at distance r is $V_c^2(r) = GM(< r)/r$, hence

$$\begin{aligned} V_c^2(r) &= \frac{GM_0}{r} + \frac{G\, M_1}{r} x^2 \quad \text{for } r \le r_1 \\ &= \frac{G(M_0 - M_1)}{r} + \frac{2GM_1}{r_1} \quad \text{for } r_1 \le r \le r_2 \\ &= \frac{G(M_0 + (2 * r_2/r_1 - 1)M_1)}{r} \;\; \text{for } r > r_2\,. \end{aligned} \qquad (11)$$

- If we neglect M_0 in the equation for V_c, then from the previous equation, $V_c^2(r = 8.5\text{kpc}) = (220\text{km s}^{-1})^2 = \frac{GM(r<8.5\text{kpc})}{r} = \frac{G(M_0 - M_1)}{r} + \frac{2GM_1}{r_1} \approx \frac{-GM_1}{r} + \frac{2GM_1}{r_1}$, for $r = 8.5$kpc. Solving for M_1 requires $M_1 = 5.98 \times 10^9 M_\odot$, hence $M(r < 8.5\text{kpc}) = 9.56 \times 10^{10} M_\odot$ which is $\gg M_0$ so our simplification is justified: the SMBH does not contribute significantly to the rotation velocity at $r = 8.5$kpc. Substituting this value for M_1 in $M_1 = 2\pi\rho_1 r_1^3$ gives $\rho_1 = 9.5 \times 10^8 M_\odot/\text{kpc}^3 = 6.44 \times 10^{-23}\text{g/cm}^3$.

- To find evidence for the super-massive black hole, it should start to contribute significantly to V_c, which starts to happen when $M_0/r \approx M_1 r/r_1^2$, ($GM_0/r$ is the contribution of the black hole to V_c^2, $M_1 r/r_1^2$ is the contribution of the 'bulge'with density distribution $\propto 1/r$ to V_c^2. Hence when these contributions are equal,

$$r = (\frac{M_0}{M_1})^{1/2} r_1 = (\frac{10^6}{5.98 \times 10^9})^{1/2}\,\text{kpc} \approx 13\text{pc}\,. \tag{12}$$

The angle θ under which you see 13pc at a distance of 5Mpc is

$$\theta = \frac{13}{5 \times 10^6} \approx 0.53\text{arcsec} \tag{13}$$

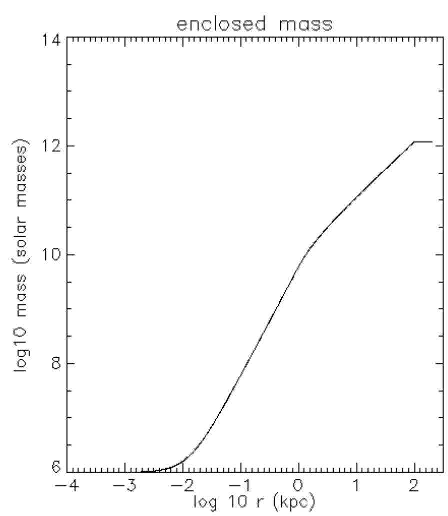

Figure 1: Enclosed mass vs radius

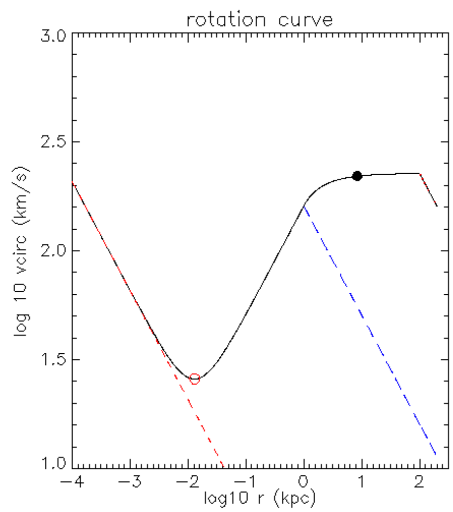

Figure 2: Rotation curve, the filled circle is at $r = 8.5$kpc, $V_c = 220$km s^{-1}. There is Keplerian fall-off $V_c \propto 1/r^{1/2}$ both very close to the black hole (where it dominates the mass), and on large scales $r \geq r_2$, where the galaxy mass is constant. In the inner region, where the mass is dominated by the $1/r$ density profile, the circular velocity rises, $V_c \propto r^{1/2}$, and where the $1/r^2$ density dominates the mass, the circular velocity becomes almost constant, $V_c \approx 220$km s^{-1}. The short dashed line is the Keplerian fall-off due to the black hole, the long-dashed is the Keplerian fall-off due to the $1/r$ density distribution. The open circle is where we computed that the black hole started contributing significantly to V_c.